Chapter 9
Geometry: Transformations, Congruence and Similarity

By the third century BCE, the Greeks had gathered together an enormous amount of geometric knowledge, based on observations from the ancient Greeks (such as Pythagoras), ancient civilizations (Babylonian, Egyptian) and their own work. Aristotle and his successors set about the task to put this knowledge on a firm logical basis. A result of their work is the "Elements of Geometry" by Euclid (the name may be of one person or of the group). Here the foundation of the subject lay in a set of "self-evident" axioms, and "constructions" by straightedge and compass. For example, two figures were called *congruent* if it we could copy one onto the other with straightedge and compass. These tools were used to copy points, line segments, circles and angles. In particular, they did not use (numerical) measure:, for example, the measure of a line segment was the distance between the pins of the compass when put at the endpoints of the segment. From there, the point of Euclid's Elements is to deduce all current knowledge from these basics, using only these axioms and tools and Aristotelian logic. It was important that the logical structure not lay in folklore and structure, but on the axioms (although it is the folklore and constructions that convince us that these axioms are self-evident). Concepts such as "same shape" and "same size" were given explicit definition, all based on a small set of "undefined objects" (point, line, etc.) whose understanding was intuitive. An objective, of course, was to minimize the number of concepts that were to be understood intuitively; while everything else is understood by definition or strict logic. The truth of assertions is justified solely by the application of logic to already known truths, and not by construction, observation, and above all, not by techniques involving movement from one place to another.

This accomplishment was monumental and formed the basis of geometric instruction for over 2000 years. We should point out that, almost immediately, philosophers began to object that (at least) one of the "self-evident" axioms was, in fact, not so self-evident. That was the axiom that dealt with parallelism. To paraphrase the problem, two lines are said to be parallel if they never meet. All other axioms could be intuitively understood by pictures and constructions on a given piece of paper (or papyrus or the sand) of finite dimensions. This axiom, however, requires us to conceive of going however far we have to, to show that two lines meet, and thus are not parallel, and worse: we can never verify that two lines *never meet*.

In the 19th century this dilemma was put to rest: there are planar geometries where all lines eventually intersect, and others for which almost all lines never intersect. These were respectively, spherical and hyperbolic geometry,. They were discovered because the applications of mathematics needed understanding of these geometries. In the late 19th century, the mathematician Felix Klein formalized a new concept of geometry, broad enough to encompass all these forms. This geometry is based on its dynamical, rather than static, use. In Kleinian geometry, the primary concept is that of *transformation*: a set of transformations are specified, and geometry becomes the study of objects that do not change under these transformations. This is the approach that is adopted in mathematical instruction of today and is called *transformational geometry*.

During the same time, other mathematical ideas were developing and maturing that would mesh with this geometric thread. The introduction of coordinates in the 17th century, and the development of linear algebra in the 19th century presented a rich set of tools in which to develop geometry: we call this *coordinate geometry* (and when we go to dimensions greater than 2, *vector geometry*. This approach provides a new perspective and leads to a complete representation of geometry in terms of algebra and through this interpretation, a new way to rediscover geometry. The word *rediscover* is used deliberately, for coordinate geometry provides us with a way to precisely calculate measures in geometry, but not a new way to develop the subject. To illustrate: distance between two points will be defined in terms of the coordinates of the points, and not in terms of a scale along a straight edge placed on the two points. That is, the Pythagorean theorem becomes the definition of length of a line segment. So, why is the Pythagorean theorem true? For the Egyptians it is an observed truth; for Euclid it follows from the (self-evident) axioms of geometry; and in transformational and coordiante geometry it becomes the basis for those approaches.

All of this will be developed in a systematic way in secondary mathematics. The objective in 8th grade is to give the students the opportunity of free exploration of the basic concepts of transformational geometry: rigid motions, dilations, congruence and similarity. In chapters 9 and 10 students will verify that rigid motions preserve the measures of line segments and angles, and that dilations preserve the measure of angles, while changing measures of line segments by a constant factor. From there we go on to observe basic geometric facts, some of which will be made explicit in the classroom, while others are discovered through the student's own work.

We take the approach that the understanding of much of secondary mathematics is dependent upon the strength of the students' geometric intuition, and that intuition is best developed through free play with the fundamental concepts and ideas. So, to some, our exposition may seem unstructured; we ask those people to go to the appendix for a structured development of transformational geometry. We hope the teacher will see it as overstructured for the 8th grade student, allowing the course of study to be developed by the class, even if it veers too much from the text.

In Chapter 2, section 2, students learned that the slope of a line can be calculated as rise/run starting with any two points on the line. To show why this works, we introduced translations and dilations, and observed their properties as transformations of the plane. We used these basic properties of dilations: it has one fixed point (the center of the dilation) and all other points are moved away (or toward) the center. There is a positive number r such that the dilation multiplies any length by r. Note that if $r = 1$, then there is no movement at all. In this case the dilation is called the *identity*: no point moves.

In this chapter we begin to look at transformations of the plane more deeply, in order to get an understanding of the shape and size of a geometric object, no matter where it is positioned on the plane. Students have already seen *shifts, flips* and *rotations*: here we reintroduce them as motions of the plane that preserve the basic geometric measures: that of angles and lengths. In discussing these motions, and dilations, one should take a dynamic, not static approach. We are not picking up an object and dropping it, we are "moving" it to its new location.

A *rigid motion* of the plane is a transformation of the plane that takes lines to lines, and preserves lengths of line segments and measures of angles. That is, under a rigid motion, a line segment and its image have the same length, and an angle and its image have the same measure. An example of a rigid motion is a *translation* (called a *shift* until now). There are two other basic kinds: *reflections* (flips) and *rotations* (turns). Now we consider two figures *congruent* (of the same shape and size) if there is a sequence of rigid motions that takes one to the other. This is a different way of looking at the equivalence of two figures without changing the meaning: if two objects are congruent by way of a Euclidean construction, then there is a sequence of rigid motions that takes one to the other. And if we can move one object onto another by rigid motions, there is a construction taking one to the other. The advantage of working with motions rather than constructions is that the idea is more directly related the the use of geometry in science and engineering: one does not put a beam on a house by construction in place, but by moving the beam from one place to the other. If we want to create a robot to do that job, we need to conceive of it in terms of rigid motions, not constructions.

In the second section we turn to dilations and scale factors: a dilation preserves lines and angles, but changes the scale of length of line segments. We say that two figures are *similar* (have the same shape) if there is a combination of rigid motions and dilations that takes one to the other.

The focus of 8th grade geometry is to explore the concepts of transformations, congruence and similarity by experimenting with them and gaining familiarity with the correspondence between constructing a new image of an object, and moving the object to its new location. We concentrate on the "what" and "how" of geometry, while high school geometry extends that basis to understanding the "why." In real-life science and industry, people almost constantly draw representations (called *graphics*) of their work, even if it is about medical procedures or finance rather than architecture or construction. In 8th grade we plant the foundations for these skills.

To begin with, students should be given an opportunity to discuss the concepts of "same shape" and "same size and shape." This is the purpose of the following example and many of the preliminary exercises in the workbook.

Figure 1 shows several sets of objects. In Figure A all the images are of the same size and shape, and we can move any one to any other one by a rigid motion. In the remaining figures, there is no rigid motion taking one figure to the other. Try to understand how to move the first object in Figure A on the others. Why can't this be done for the other sets of objects? Note that in Figure B, the figures are of the same shape, but not of the same size, and in Figures C and D the figures are neither of the same size nor shape.

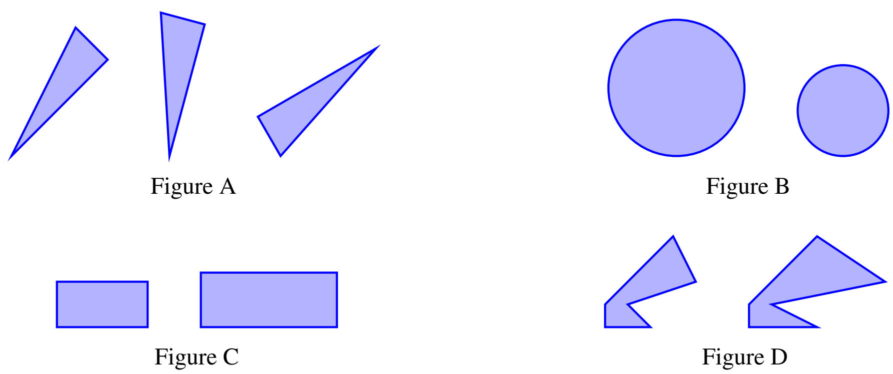


Figure 1

Let's recall some basic geometric facts that have been observed in previous grades.

1. A line is determined by any two different points on the line, by placing a straight edge against the points and drawing the line.

2. Two lines coincide (are the same line) or intersect in precisely one point or do not intersect at all. The issue may come up: what if they do not intersect on my paper, how do I know whether or not they ever intersect? Because the question did come up in the days of Euclid, it generated a controversy that lasted for almost 2000 years.

3. Two circles do not intersect, or intersect in a point, or intersect in two points. If they intersect in more than two points, they actually coincide.

4. Two lines that do not intersect are said to be *parallel*. If two lines intersect and all the angles at the point of intersection have the same measure, the lines are said to be *perpendicular*.

5. The sum of the lengths of any two sides of a triangle is greater than the third.

Section 9.1: Rigid motions and Congruence

Understand congruence in terms of translations, rotations and reflections, (rigid motions) using ruler and compass, physical models, transparencies, geometric software.

Verify experimentally the properties of rotations, reflections and translations:

a) lines are taken to lines, and line segments to line segments of the same length;

b) angles are taken to angles of the same measure;

c) parallel lines are taken to parallel lines. 8G1

A rule that assigns, to each point in the plane another point in the plane is called a *correspondence*. Often a correspondence is defined in terms of coordinates, and written this way: $(x, y) \rightarrow (x', y')$, where the values of x', y' are given by the rule, which may be a formula or a set of instructions. In the case of a formula, we call it the *coordinate rule*

Example 1.

Where possible give the coordinate rule for the correspondence.

- **a.** Move a point P to a point P' on the same ray through the origin that is twice the distance from the origin.
- **b.** Move every point in the plane 3 units to the right and 1 unit down.
- **c.** Multiply the first coordinate by 2 and the second coordinate by 3.
- **d.** Replace each coordinate of the point by its square.
- **e.** Interchange the two coordinates. How do we describe the transformation in geometric terms?

Solution.

- **a.** If (x, y) are the coordinates of the point P, then the coordinates of any point on the ray from the origin through P is of the form (rx, ry) for some $r > 0$ (for a line through the origin represents a proportional relationship). In our case $r = 2$, so the coordinate rule is $(x, y) \rightarrow (2x, 2y)$.

- **b.** If (x, y) are the coordinates of a point P, then the coordinates of the point three units to the right and one unit down are $(x + 3, y - 1)$. So the coordinate rule is $(x, y) \rightarrow (x + 3, y - 1)$.

- **c.** Here the coordinate rule is $(x, y) \rightarrow (2x, 3y)$.

- **d.** The coordinate rule is $(x, y) \rightarrow (x^2, y^2)$.

- **e.** The coordinate rule is $(x, y) \rightarrow (y, x)$. This can be described as a flip (reflection) in the line $y = x$.

A *mapping* (or *transformation*) T of the plane is a correspondence that has the property that different points go to different points; that is, for two points $P \neq Q$ we must also have $T(P) \neq T(Q)$. This is in fact just what a map does: it takes a piece of the surface of the earth and represents it, point for point, on the map M. When we study the effect of a mapping on objects, it is useful to call the object K the *pre-image* and the set of points to which the points of K are mapped is the *image*, denoted $T(K)$. Of the rules described in example 2, rules 1,2,3,6 are mappings, while rules 4 and 5 are not. Before going on, a little more vocabulary. An object is *fixed* under a mapping, if the mapping takes the object onto itself. When we say that an attribute is *preserved* we mean that if the original object has that attribute, so does the image object.

To be useful, geometrically, a transformation must preserve features of interest: for example, a scale drawing of an object changes only the dimensions, and not the shape, of the object. In this section we are interested in mappings that preserve both the dimension and shape of objects. These are the *rigid motions*. These are mappings of the plane onto itself that takes lines to lines and preserve lengths of line segments and measures of angles. That definition is the starting point of geometry in Secondary 1, but not in 8th grade, where the emphasis is on an intuitive understanding of rigid motions and their action on figures. So, for us, rigid motions are introduced by visualization and activities. Take two pieces of transparent paper with a coordinate grid. Place one on top of the other so that the coordinatizations coincide. A *rigid motion* is given by a motion of the top plane that does not wrinkle or stretch the piece of paper.

Given a figure on the plane, we can track its motion by a rigid motion T in this way. Place a piece of transparent graph paper on top of another so that the coordinate axes coincide. We start with a figure on the bottom piece of paper. Copy that figure on the upper piece of paper. Enact the rigid motion T by moving the upper piece of paper. Now trace the image on the upper piece of paper onto the lower. This figure is the result of moving the given figure by the rigid motion T. Keep in mind that the lower piece of paper is where the action is taking place; the upper piece is the action. Students should experiment with these motions using transparent papers. In the process they will observe that these motions do preserve lines and the measures of line segments and angles. If a figure doesn't change; that is T takes the figure onto itself, we say that the figure is *fixed*. Students may observe that there are three fundamental kinds of rigid motions characterized by: a) no point is left fixed b) precisely one point is left fixed, c) there is a line all of whose points are left fixed.

Rigid motions include

- **Translation**: these are the rigid motions T of the plane that preserve "horizontal" and "vertical": that is, horizontal lines remain horizontal, and vertical lines remain vertical. In fact the image of *any* stays parallel to the given line. We can describe this as sliding the top piece of paper over the bottom so that the horizontal and vertical directions remain the same.
- **Reflection** in a line. Select a line on the plane, this will be the *line of reflection*. The line separates the plane into two half-planes. Lines perpendicular to the line of reflection go to themselves. For a point on one of the half planes, its image under the reflection is in the other half-plane, on the same line perpendicular to the line of reflection, and of the same distance from the line of reflection. We effect this with the transparent paper in this way: fold the top transparency (in three dimensions) along that line so that the sides determined by the line exchange places, and so that nothing on the line moves.
- **Rotation** about a point: these are the rigid motions of a plane that leave one point fixed. This point is called the *center of the rotation*. The center will not move. All other points are moved along a circle with that point as center. The angle between any ray through the center and its image is constant, called the *angle of reflection*.

Through experimentation with these movements, students should observe that translations do not leave any points fixed; rotations are rigid motions that leave just one point fixed; for a reflection, all the points on the line of reflection remain fixed. In the next subsections we will work with these motions in detail and collect together their properties.

Example 2.

Of the rules in Example 1, which can be represented by a rigid motion?

Solution.

a. This is not a rigid motion: objects change size. Furthermore, it is a transformation that has one fixed point, but it is not a rotation. In particular. it cannot be illustrated using transparent papers, for it stretches the top paper.

b. This can be realized as a rigid motion: shift the top piece of paper so that the the origin goes to the point $(3, -1)$.This is an example of a translation.

c. This is like a) but even more complicated: the horizontal stretch is 2, and the vertical stretch is 3.

d. The squaring rule does not take distinct points to distinct points: for example (1,1) and $(-1, -1)$ both go to (1,1). In fact, for any positive a and b, all four points (a, b), $(-a, b)$, $(a, -b)$, $(-a, -b)$ go to the same point (a^2, b^2).

e. This transformation is realized by reflection in the line $y = x$.

Translations

We have defined *translation* as a rigid motion of the plane that moves horizontal lines to horizontal lines and vertical lines to vertical lines. Let's find the coordinate rule for a translation. Starting with a particular translation, let (a, b) be the coordinates of the image of the origin. We want to show that the pair of numbers (a, b) completely determines the translation; in fact, the coordinate rule is: $(x, y) \to (x + a, y + b)$. For a dynamic visualization, go to `http://www.mathopenref.com/translate.html`.

For any point (x, y), draw the rectangle with horizontal and vertical sides with one vertex at the origin and the other at (x, y) (see Figure 2). The translation takes this rectangle to a rectangle with horizontal and vertical sides with one vertex at (a, b) and the other at the image of (x, y). Since the lengths of the sides are preserved, the image rectangle has the same dimensions as the pre-image, and so the vertex across the diagonal from (a, b) has to be $(x + a, y + b)$. But that is the image of (x, y).

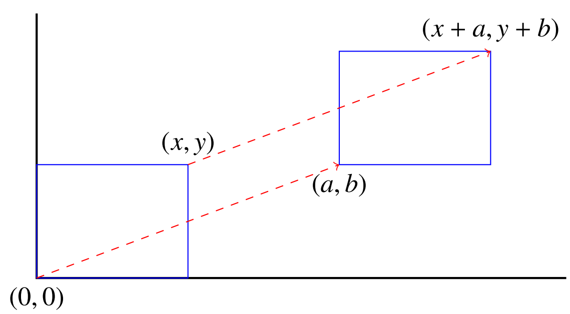


Figure 2

We refer to (a, b) as the *vector* for the translation, for it shows both the direction in which any point is translated, and also the distance it s translated. Now, in our figure, the point (x, y) was chosen to be in the first quadrant; but the same reasoning works for any point (x, y).

This reasoning also shows that a translation preserves the slopes of lines (in particular, any line and its image are parallel). In Figure 3 we show the effect of a translation on the line L, denoting its image by L'. Draw the slope triangle AVB for the line L as shown. Now, the translation moves that triangle to the triangle $A'V'B'$ as shown. Since the translation preserves "horizontal" and "vertical," $A'V'B'$ is a slope triangle for the line L'. Now since the translation preserves lengths, the horizontal and vertical legs of the slope triangle on L' have the same lengths as the horizontal and vertical legs of the slope triangle on L, so the lines have the same slope. Figure 3 is deliberately drawn with L' not parallel to L so as to show why that assumption is incorrect.

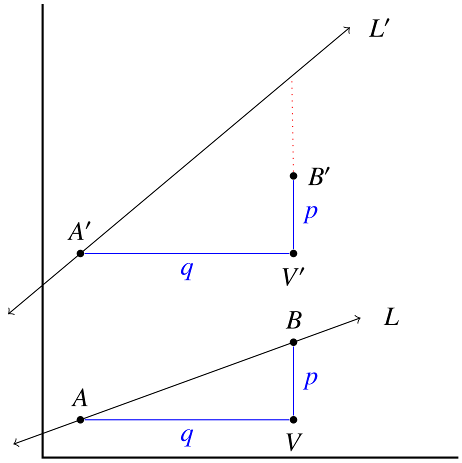


Figure 3

Properties of translations

- A translation preserves the lengths of line segments and the measures of angles.
- For a translation, there is a pair (a, b), called the *vector* of the translation, such that the image of any point (x, y) is the point $(x + a, y + b)$.
- Under a translation, the image of a line L is a line L' parallel to L. Furthermore, translations take parallel lines to parallel lines. This is because a translation does not change the slope of a line.
- A translation that does not leave every point fixed does not leave any point fixed.

Reflections

We have defined a *reflection* by the action of flipping (or folding) along a line L, called the *line of reflection*. A reflection can be described this way: it is a motion that leaves every point on L fixed, and for a point P not on L, with P' its image under the reflection, L is the perpendicular bisector of the line segment PP'. We now show that reflections as we visualize them (folding the plane along the line L) have these properties. For a dynamic visualization, go to `http://www.mathopenref.com/reflect.html`.

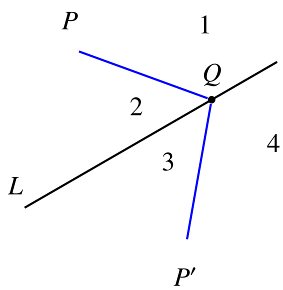


Figure 4

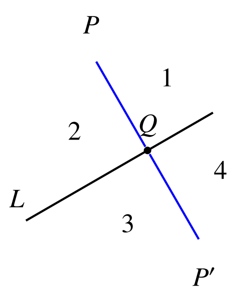


Figure 5

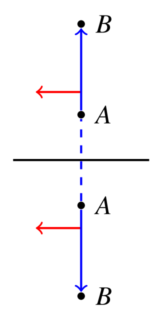


Figure 6

First, it is clear, since it is described by a motion that does not stretch our paper in any direction, that a reflection preserves the lengths of line segments and the measure of angles. Let L be the line of reflection for the reflection

T, let P be a point on one side of L and P' the image of P under the reflection. Draw any line from P to a point Q on L. Let P' be the image of the point P under the reflection. This situation is depicted in Figure 4.

Since a reflection takes lines to lines and leaves the point Q fixed, the image of the segment PQ is the segment $P'Q$. Since the lengths of these segments are the same, and Q was chosen as any point on L, we conclude that the points P and P' are at the same distance from any point on the line L. Furthermore the image of $\angle 1$ is $\angle 4$, so they have the same measure, and the image of $\angle 2$ is $\angle 3$, so they have the same measure. In particular, if Q is chosen so that PQ is perpendicular to L (and thus all angles at Q are the same), since we already know that the segments PQ and $P'Q$ have the same length, we conclude that L is the perpendicular bisector of PP'. See Figure 5.

Something special happens with reflections that does not happen with other motions. Notice that for translations and, as we will see, rotations, we do not have to lift the top piece of transparent paper off the bottom piece; but with reflections we must do so; we execute what we can call a *flip* over the line of the reflection. This has an important effect that is not shared with translations and rotations. Suppose that A and B are two different points on a line perpendicular to the line of reflection L. Consider that line as directed from A to B, and at A draw a little line segment on the left side of the directed line segment AB. Now reflect this configuration in the line L. The little line segment now lies on the right of the directed image line segment $A'B'$ (see Figure 6). Another way of putting this is that the rotation from the direction of AB to the direction of the small red arrow is counterclockwise in the original, but clockwise in the image. Since the point in question could have been any point in the plane, what has happened is that the reflection changed the sense of clockwise to counterclockwise everywhere. This sense of rotation, clockwise or counterclockwise, about any point is called *orientation*. In short, a reflection changes the orientation of the plane.

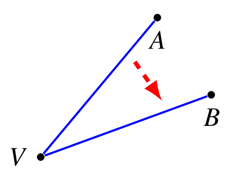


Figure 7A

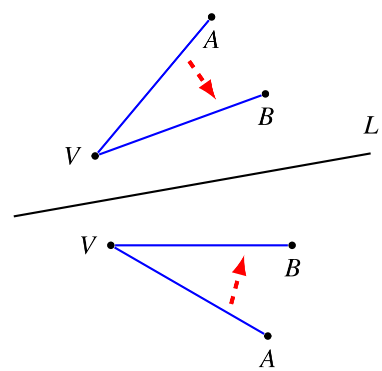


Figure 7B

Orientation can also be described in terms of angles. Consider an angle $\angle AVB$ (with vertex V) determined by the rays VA and VB. Looking out at the angle from V we can say that one of the rays is *clockwise* from the other (in Figure 7A, the ray VB is clockwise from VA). The reverse direction is called *counterclockwise*. Now the point we want to make is that reflections interchange the rays of an angle in the sense of orientation. This is depicted in Figure 7B, representing a reflection in the line L. The angle $\angle AVB$ goes to the angle $A'V'B'$ under the reflection. But, while the ray VB is clockwise to the ray VA, the image ray $V'B'$ is counterclockwise to the image ray $V'A'$. This is what happens when we look in a mirror: our left hand is on the right side of that person in the mirror.

Properties of Reflections:

- **a.** A reflection preserves the lengths of line segments and the measures of angles.
- **b.** For a reflection, there is a line L, called the *line of the reflection*, such that for any point P, L is the perpendicular bisector of the line segment joining P to its image.
- **c.** A reflection leaves every point on L fixed, and interchanges the two sides of that line. If the image of a point P is P', then, for any point Q on L, the line segments PQ and $P'Q$ are the same.
- **d.** A reflection reverses orientation; that is if two rays start at the same point, and ray 2 is clockwise from ray 1, the the image of ray 2 is counterclockwise from that of ray 1.

It is possible to describe all rigid motions by coordinate rules; at this time it is most useful to just do this for particular special cases. Do so for these particular reflections: in the coordinate axes and in the lines $y = x$ and $y = -x$.

Example 3.

Find the coordinate rule for the reflections **a.** the x-axis, **b.** the y-axis, **c.** the line $y = x$, **d.** the line $y = -x$.

Solution.

a. Reflection in the x-axis leaves the x coordinate the same and changes the sign of the y coordinate. For this reflection takes vertical lines to vertical lines, and so the x-coordinate is fixed. For any point (x, y) (with $y \neq 0$), its image is a point (x, y') with $|y| = |y'|$ and $y \neq y'$; the only possibility is that y' is $-y$.

b. Reflection in the y-axis leaves the y coordinate the same and changes the sign of the x coordinate. This has the same argument as for part **a**.

c. Reflection in the line $L : y = x$ exchanges the coordinates: a point (x, y) goes to the point (y, x). To show this, let's start with a point (a, b) not on the line, and thus $a \neq b$. See Figure 8 for the setup. Now draw the horizontal and vertical lines trom (a, b) to the line L. The horizontal line ends at (b, b) and the vertical line at (a, a). When this triangle gets reflected in the line L, the sides of the image line will consist of a vertical line segment from (b, b) and a horizontal line segment fro (a, a). The point of intersection of these lines has the coordinates (b, a) and is the image point of (a, b).

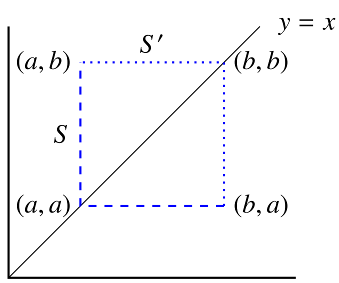


Figure 8

d. Using the same argument, but being super careful about signs, we can show that reflection in the line $L : y = -x$ can be described in coordinates as $(x, y) \rightarrow (-y, -x)$.

Rotations

We have defined a *rotation* as a rigid motion that turns a figure about a fixed point, called the *center* of the rotation. Since lines are mapped into lines and the center C is fixed, any ray with endpoint C is moved to another ray with endpoint C. A rotation can be defined by this property: the angle between any ray with endpoint C and the image of that ray always has the same measure α.

To see this, begin with FIgure 9: C is a point on the plane, and we are considering a rotation R with center C. Let A be a point on the horizontal ray from C to the right, and draw its image point A". As in the figure, denote the angle between the rays CA and CA' by double arcs. This is the angle of the rotation. Now take another typical point B, and denote its image point B'. Then the angle between the ray CB and CB' is also the angle of rotation, so it can be indicated by the double arch. Label the angles $\angle 1,\ \angle 2,\ \angle 3$ a shown in Figure 9. What we have seen is that

$$\angle 1 + \angle 2 = \angle 2 + \angle 3,$$

so $\angle 1 = \angle 3$. But $\angle 1$ is the angle between the ray CA and CB and $\angle 3$ is the angle between the ray CA' and CB', so is the image angle , and thus we have shown that rotations preserve the measure of angles.

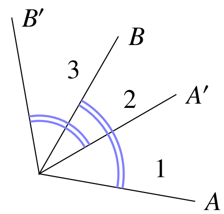


Figure 9

For a dynamic visualization of this discussion, go to http://www.mathopenref.com/rotate.html.

Properties of rotations:

- A rotation preserves the lengths of line segments and the measures of angles.
- For a rotation, there is a point C, called the *center of the rotation*, and an angle α called the *angle of rotation*. For any point P with image Q, the angle $\angle PCQ = \alpha$.
- A rotation preserves orientation; that is, if two rays start at the same point, and the second is clockwise from the first, then the image of the second is also clockwise from the first.

Example 4.

Find the coordinate rule for a rotation of

a. 90° counterclockwise,

b. 90° clockwise (denoted by -90°),

c. 180° and **d.** -180°.

Solution.

a. See Figure 10, where P is a point in the first quadrant and Q is the image of P. The rotation moves triangle I into the position of triangle II. Note that the horizontal leg of triangle I corresponds to the vertical leg of triangle II, and the vertical leg of triangle I corresponds to the horizontal leg of the image, triangle II. The lengths of corresponding line segments are the same, but since the image is in the second quadrant the first coordinate is negative so we must have $(x, y) \to (-y, x)$ as labeled. This argument works as well no matter in which quadrant we start with the point P.

b. The same argument works, only now use Figure 11, and we conclude that the coordinate rule for a clockwise rotation by a right angle (of - 90°) has to be $(x, y \to (y, -x)$

Take a moment to note that a rotation by 90° takes a line into another line perpendicular to it. We saw in Chapter 2 that the product of the slope of a line and that of its image under a rotation by 90° is -1. Note

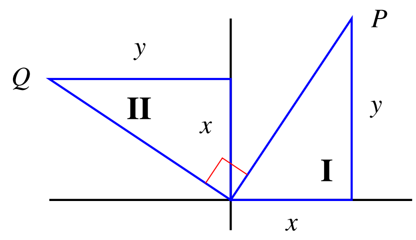


Figure 10

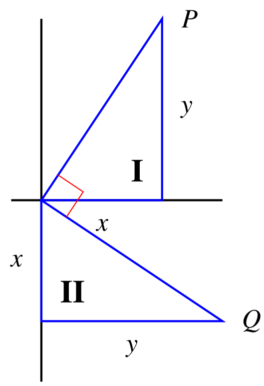


Figure 11

that this is demonstrated in Figures 10 and 11: the slope of the original line is y/x and that of its image is $-x/y$.

c. See Figure 12, where P again is in the first quadrant and Q is its image. Since the rotation is by 180°, P and Q lie on the same line through the origin, and the length of the segment CP and CQ are the same. In other words, Q is diametrically opposite to P, so is the point $(-x, -y)$.

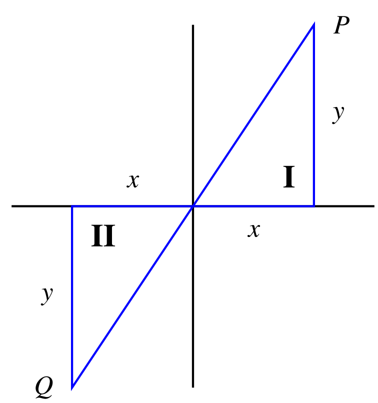


Figure 12

Another way of seeing this is to recognize that a rotation by 180° is a rotation by 90° followed by another rotation by 90° . Now rotation by 90° interchanges the coordinates and puts a minus sign in front of the first one. Thus the succession of two 90° rotations can be written in coordinates by

$$(x, y) \to (-y, x) \to (-x, -y) .$$

d. The same argument holds for a rotation by -180° , so is also given by the coordinate rule $(x, y) \to (-x, -y)$.

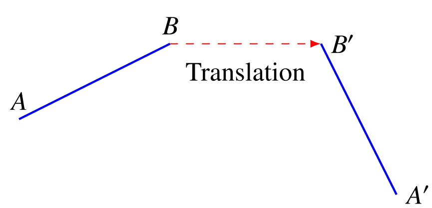
Extension. Succession of rigid motions

A rigid motion is a transformation of the plane that takes lines into lines and that preserves lengths of line segments and measures of angles. If we follow one rigid motion by another,we get a third motion which clearly has the same properties: lines go to lines and measures of line segments and angles do not change. We have discussed specific kinds of rigid motions: translations, reflections and rotations. It is a fact that every rigid motion can be viewed as a a succession of motions of one or more of these types; in this section we will look at such examples.

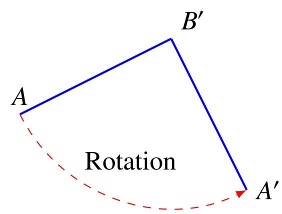
Example 5.

Given two line segments AB and $A'B'$ of the same length, there is a rigid motion that takes one onto the other.

Solution. Figure 13A shows the two line segments, and indicates that we can translate the point B to the point B', getting the picture in Figure 13B. Now rotate the line segment AB' around the point B' through the angle $\angle AB'A'$ so that the segments AB' and $A'B'$ lie on the same ray. But since the segments have the same length, the point A lands on A', and the succession of the translation by the rotation is the rigid motion taking AB to $A'B'$.



13A



13B

Example 6.

Given two circles of the same radius, there is a rigid motion, in fact, a translation, taking one circle onto the other. In fact, the translation of one center to the other does the trick. Can you explain why? Hint: use the definition of “circle.”

Solution. Let r be the radius of the given circle, and C its center. Let C' be the center of the second given circle. The translation from C to C' preserves distance, so all points of distance r from the first circle go to points of distance r from the second.

Example 7.

If two angles have the same measure, there is a rigid motion of one to the other.

Solution. Let AVB and $A'VB'$ be the two angles, as in Figure 14.. We can arrange that they have the same vertex; for if not, we can translate one vertex to the other, bringing us to Figure 14. We have already taken another liberty: we have labeled the rays of each angle so that the orientation is consistent: VB is clockwise from VA and VB' is clockwise from VA'. (If this wasn't the case originally, how can we make it so?) Now rotate with center V so that the ray VA lands on the ray VA'. Since the original angles had the same measure, and we have set up the orientation correctly, the ray VB falls on the ray VB'. The combination of the translation and rotation is the rigid motion landing one angle onto the other.

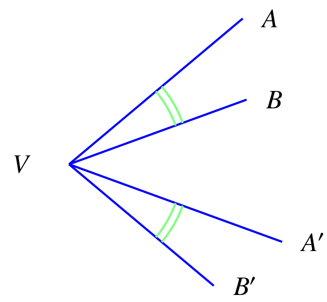


Figure 14

Example 8.

Under what conditions can we find a rigid motion of one rectangle onto another?

Solution. First of all, rigid motions preserve lengths and angles, so any rigid motion will always move a rectangle to another rectangle whose side lengths are the same. If there is a rigid motion of rectangle R onto rectangle R', the lengths of corresponding sides must be the same.

To answer the question: two rectangles are congruent if the lengths of corresponding sides are the same. if this condition holds, then there is a rigid motion of one rectangle onto the other. We will show this using Figure 15 of two rectangles $ABCD$ and $A'B'C'D'$ with corresponding sides of equal lengths. We have labeled the vertices so that the routes $A \to B \to C \to D$ and $A' \to B' \to C' \to D'$ are both clockwise. (Can you check that we can really do that?). By Example 7, we can find a rigid motion taking the angle DAB onto the angle $D'A'B'$ (translate A to A' and then rotate). Since the lengths of corresponding sides are equal, that tells us that D lands on D' and B onto B'. Since both figures are rectangles, that forces C onto C', so the rectangles are congruent.

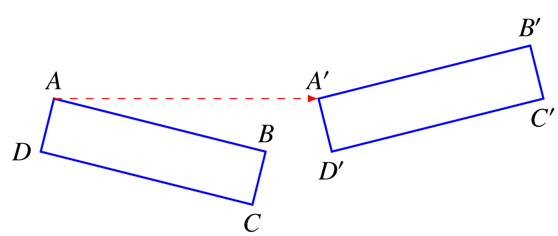


Figure 15

Example 9.

Since reflections reverse the orientation on the plane, the succession of two reflections preserves orientation, so has to be a pure rotation or a pure translation or a combination of the two. How do we know when we get a translation or when we get a rotation?

Solution. Let R be the reflection in the line L, S the reflection in the line L', and T the combined motion: R followed by S.

a. If the lines L and L' are the same line, then S just undoes what R did, and the succession of R by S leaves every point where it is. In this case, the succession of reflections is the *identity* motion - no motion at all.

b. Suppose that the lines L and L' are different, and intersect in a point C . Then C is fixed point for each, so is fixed under the succession T. Suppose T had another fixed point P. Then the first reflection (R) sends P to another point P' and S returns P' to P. Well, that means that P and P' are reflective images in both L and L', and that can only be if the lines are the same. Since they are not, T has only one fixed point: C. But the only motions with one fixed point are the rotations.

c. If L and L' are different lines and have no point of intersection, then they are parallel. In this case, the transformation T formed by the succession of the two reflections has no fixed points. For if $T(P) = P$, this tells us that S takes $R(P)$ back to P, but the only reflection that does that is R. Since the lines are different, $S \neq R$, so there is no point P such that $T(P) = P$; that is there are no fixed points. Only translations are without fixed points, so T is a translation.

Example 10.

Let R be the reflection in the y axis, and S, reflection in the x axis.

a. Describe the rigid motion T defined by the succession: R followed by S.

b. Do the same problem with either of the lines of reflection replaced by the line $y = x$.

Solution.

a. Reflection in the y-axis changes the sign of the first coordinate, and reflection in the x-axis changes the sign of the second coordinate.The effect of both is to change the sign of both coordinates. The coordinate rule, then, for R followed by S (or the other way around) is $(x, y) \to (-x, -y)$

b. Let's denote reflection in the line $y = x$ by U.. That effect is to interchange coordinates. So, R followed by U is given in coordinates by: $(x, y) \to (-x, y) \to (y, -x)$, which is rotation by -90°. U followed by S is, in coordinates, given by: $(x, y) \to (y, x) \to (y, -x)$. The same answer!

Suppose we interchange the orders of R, S and U, do the answers change?

Example 11.

Let R be reflection in the line $x = 1$, and S reflection in the line $x = 2$. Describe the translation T, the succession R followed by S.

Solution. Both R and S take vertical lines to vertical lines, and take horizontal lines to themselves, so that is true of the succession T. R takes (0,0) to (2,0) and S leaves (2,0) alone, so T takes (0,0) to (0,2). Since a translation does to all points what it does to one, we can say, in coordinates, that T takes (x, y) to $(x + 2, y)$.

Properties of a succession of two rigid motions:

- Rotations: if they have the same center then the succession of the two is a rotation with that center, and whose angle is the sum of the angles of the two given rotations.
- Translations: if T is a translation by (a, b), and T' the translation by (a', b'), then the succession of one after the other is the translation by $(a + a', b + b')$.
- Reflections:

  a. If the lines of the reflection are the same, we get the identity (that is, there is no motion: every point stays where it is.

  b. If the lines of the reflection are parallel, we get a translation.

  c.If the lines of the reflection intersect in a point, we get a rotation about that point. **End Extension**

Congruence

Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them. 8G2

Two figures are said to be *congruent* if there is a rigid motion that moves one onto the other. In high school mathematics the topic of congruence will be developed in a coherent, logical way, giving students the tools to answer many geometric questions. In 8th grade we are much more freewheeling, discovering what we can about congruence through experimentation with actual motions. In this section we will list some possible results that the class may discover; many classes will not discover some of these, but instead discover other interesting facts about congruence.

EXAMPLE 12.

Using transparencies, decide, among the four triangles in Figure 16 , which are congruent.

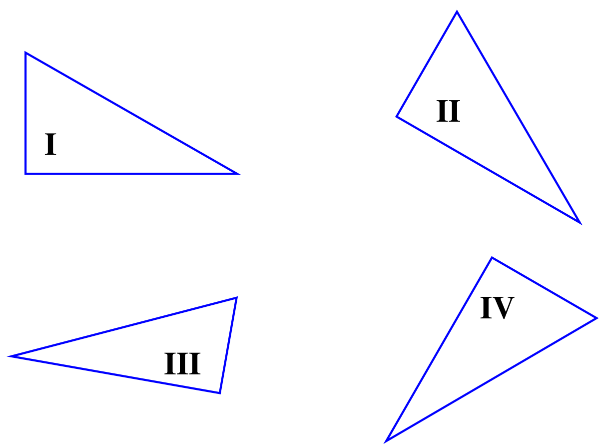


Figure 16

SOLUTION. Translating and then rotating appropriately, we can put triangle I on top of triangle II, so these are congruent. When we do the same, moving triangle I to triangle III, we find that the shortest leg of triangle I is shorter than the shortest leg of triangle II; since the short legs must correspond, these triangles are not congruent. As for triangle IV, we can translate the right angle of triangle I to that of triangle IV. Since both short legs are vertical, we see they coincide. If we now reflect in that short leg, we land right on triangle IV. Thus, I and IV are also congruent.

- All points in the plane are congruent.

This may seem unnecessary to point out, but it does state a fact: given any points P and Q in the plane, there is a rigid motion taking P to Q. Actually, there are many. First of all, there is the translation of P to Q, and we can follow that by any rotation about the point Q. There is also a reflection of P to Q: fold the paper in such a way that P lands on top of Q. Then the crease line of the fold is the perpendicular bisector of the line segment PQ, and reflection in this line takes P to Q.

The following have been observed in the preceding section:

- Two line segments are congruent if they have the same length; otherwise they are not.
- Two angles are congruent if they have the same measure.

- Two rectangles are congruent if the side lengths of corresponding sides are the same. In particular, all squares of the same area are congruent, but not all rectangles of the same area are congruent.

Finding criteria for the congruence of triangles and demonstrating that those criteria imply congruence is a major topic in grade 9 mathematics. The intent in grade 8 is to find criteria for congruence through exploration, and provide some reason for coming to the conclusion for congruence Let us illustrate with the theorem known as SSS: given two triangles, if corresponding sides have the same lengths, then the triangles are congruent.

Example 13.

Find a reason why SSS might be true.

Solution. Recall this figure (Figure 17) from Chapter 5 of the Grade 7 text. At that time it was used to explain the "triangle inequality," here we look at it as showing that a triangle with given side lengths is "unique."

Let's say we have a triangle ABC of side lengths a, b, c. By a rigid motion, we can assume that the side of lengt c is a horizontal segment, with endpoinst A and B (as in Figure 17). Now, draw a circle with center B and radius length a (indicated by the dashed curve in Figure 17). Since the side BC of the triangle is of length a, the vertex C lies on the dashed circle. There are only two points that are possible: the point designated as C, or its reflection in the line through AB, designated by C'. Now, suppose we do the same thing with another triangle $A'B'C'$ of the same side lengths. Then the line segment $A'B'$ coincides with AB. The vertex C will have to fall on the dashed circle, and since the the side length $b' = b$, C' has to land on C or D. Since the triangles ABC and ABD are congruent (by the rigid motion of reflecton in the line through AB), the triangles ABC and $A'B'C'$ are congruent.

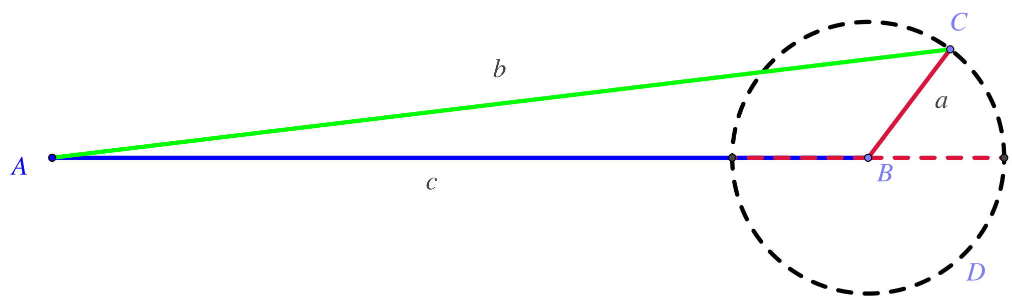


Figure 17: The uniqueness of a triangle with given side lengths

This argument is not the only one possible; that is why the example asks: "Find **a** reason . . . ."

Section 9.2: Dilations and Similarity

Understand similarity in terms of rigid motions and dilations using ruler and compass, physical models, transparencies, geometric software. 8G3,4

Properties of dilations

Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates 8G3.

Verify that dilations take lines into lines, takes parallel lines to parallel lines and that a line and its image under a dilation are parallel.

Recall that in chapter 2, in the section on the slope of a line, we introduced the idea of a dilation in order to show that the slope of a line can be calculated using any two points. Let's start by reviewing that discussion.

A *dilation* is given by a point C, the *center* of the dilation, and a positive number r, the *factor* of the dilation. The dilation with center C and factor r moves each point P to a point P' on the ray CP so that the ratio of the length of image to the length of original is r: $|CP'|/|CP| = r$.

Note that if $r = 1$, nothing changes; this "dilation" is called the *identity*. If $r > 1$, everything expands away from the center C, and if $r < 1$ everything contracts toward C.

The important fact about a dilation is that, for *every* line segment, the length of its image is r times the length of the segment. For students at this point, this will be easy to verify by experimentation, but the reason is not so obvious. In the next chapter we'll see this follows from the Pythagorean theorem; but it is also a fact of Euclidean geometry, as will be shown in the appendix.

Dilations are also directly connected to scale changes. Suppose that we want to plan out the configuration of houses and open space on a particular city block. To start, we draw a scale model of the block: the upper rectangle in Figure 18. After some work, we decide that we need a scale drawing that is 50% larger. This gives us the lower rectangle in Figure 18. As drawn these two rectangles are related by a dilation. Specifically, draw the line through the upper right vertices of the two rectangles. This line intersects the vertical line through the left sides of the two rectangles at some point C. Now, the dilation with center C and factor 1.5 takes the upper rectangle onto the lower one.

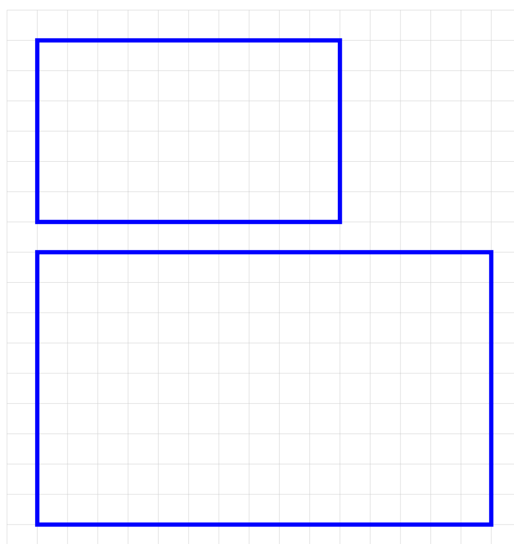


Figure 18

Given a coordinate grid, all dilations can be represented algebraically in the coordinate variables. Here let's provide that representation in the special case that the center of the dilation is the origin. Suppose the factor of the dilation is r. Then the length of any interval is multiplied by r, so that the point $(x, 0)$ goes to the point $(rx, 0)$ and the point $(0, y)$ goes to the point $(0, ry)$. It follows (see Figure 19) that $(x, y) \to (rx, ry)$.

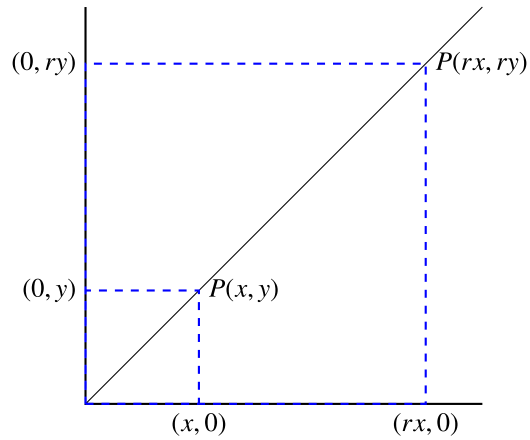


Figure 19

Properties of the dilation with center C and factor r:

- If P is moved to P', then $|CP'| = r|CP|$.
- If P is moved to P' and Q is moved to Q', then $|Q'P'| = r|QP|$.
- The dilation takes parallel lines to parallel lines.
- A line and its image are parallel.
- An angle and its image have the same measure.

All of these facts, except the last, were explored through examples in Chapter 2. Let's complete by showing that an angle and its image under a dilation are actually congruent. In Figure 20, suppose that $\angle A'V'B'$ is obtained from $\angle AVB$ by a dilation. First of all, corresponding lines are parallel. Now, translate V to V'. Since corresponding lines under a translation are parallel, the ray VA must go to the ray $V'A'$ and the ray VB to the ray $V'B'$. Thus the translation T takes the angle $\angle AVB$ to the angle $\angle A'V'B'$ so they have the same measure.

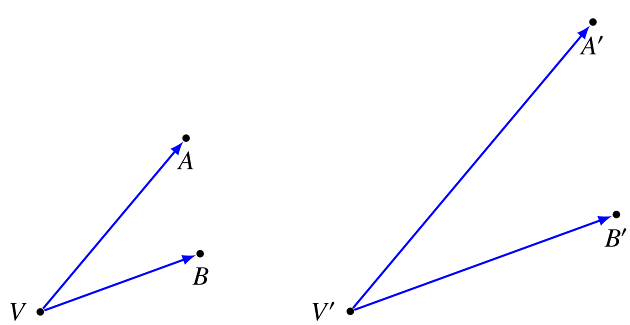


Figure 20

Similarity

Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them. 8G4

Two figures are said to be *similar* if there is a sequence of rigid motions and dilations that takes one figure onto the other. For example, the rectangles in Figure 17 are similar, since there is a dilation with center the origin that moves one to the other. Note that, the dilation with center C and factor r is undone by the dilation with center C and factor r^{-1}.

Let's collect together facts about similarity that have already been discussed or are easy to verify.

Similar Figures

- Congruent figures are similar. Two figures are congruent if there is a sequence of rigid motions placing one on the other. That sequence is certainly "a sequence of rigid motions and dilations," so is a similarity.
- Any two points or angles are similar because they are congruent.
- Any two line segments or rays are similar. Let AB and CD both be either line segments or rays. Translate A to the point C, and then rotate the image of AB so that it and CD lie on the same line.

  Let's take the case of rays. Either the rays coincide, or they form the two different rays of the same line. In the second case, another rotation by 180° makes the rays coincide.

  Next look at two line segments. Move the segment AB, by rigid motions as above, so that A falls on C, and AB and CD now lie on the same ray starting from C. Let $r = |CD|/|AB|$. Then the dilation with center C and factor r places AB on top of CD.
- Any two circles are similar. Let C be the center of one of the circles and R its radius; and C' the center of the other, and R' its radius. Translate C to C'. Now the circles are concentric. Let $r = R'/R$. Then the dilation of factor r places the first circle on top of the other.
- For two similar triangles, the ratios of corresponding sides are all the same. This is because rigid motions do not change lengths, and dilations change all lengths by the same number, the factor of the dilation.
- For two similar triangles, the measure of corresponding angles is the same. This is because rigid motions and dilations do not change the measure of angles.
- If corresponding angles of two triangles have the same measure, then the triangles are similar.

The last statement is one that can easily - and should be- observed in Grade 8. In Grade 10, students will learn the *Fundamental Theorem of Similarity*: Two triangles are similar if and only if the measures of corresponding sides are in the same ratio (what is the same, are proportional. In the following extension, we provide an informal demonstration of this fact.

Extension

EXAMPLE 14.

If corresponding sides of triangles ABC and $A'B'C'$ are parallel, then the triangles are similar. First, translate A to A'. Since a translation takes a line into a line parallel to it, the line of AB is moved to the line of $A'B'$ and the line of AC is moved to the line of $A'C'$, and the image of AB is parallel to $A'B'$, giving us the picture shown in Figure 21.

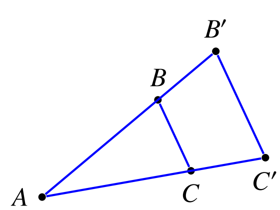


Figure 21

For $r = |AB'|/|AB|$, the dilation with center A and factor r takes B to B' and the segment BC to a parallel segment starting at B'. But, by hypothesis, $B'C'$ is parallel to BC, so is the image of BC under that dilation. Thus the dilation takes ΔABC onto $\Delta A'B'C'$.

EXAMPLE 15.

If corresponding angles of triangles ABC and $A'B'C'$ have the same measure, the triangles are similar.

SOLUTION. Since any two angles of the same measure are congruent, we can find a sequence of rigid motions that takes $\angle CAB$ onto $\angle C'A'B'$. This puts us in the situation of Figure 20, except that we do not know that the segments BC and $B'C'$ are parallel. But we do know (it is part of the hypothesis) that $\angle ABC$ and $\angle AB'C'$ are equal. That is enough; apply the dilation with center at A and factor $r = |AB'|/|AB|$. This takes B to B', and since the angle doesn't change, the ray BC lands on the ray $B'C'$. So the image of C under that dilation lies on the line of $B'C'$ and the line of AC', so it has to be C'. Thus the first triangle lands on top of the second triangle, and they are similar,

Summary

For easy reference, we gather together the coordinate rules of particular rigid motions and similarities.

- The coordinate rule for a translation by the vector (a, b) is $(x, y) \to (x + a, y + b)$.
- The coordinate rule for the reflection in the x-axis is $(x, y) \to (x, -y)$.
- The coordinate rule for the reflection in the y-axis is $(x, y) \to (-x, y)$.
- The coordinate rule for the reflection in the line $x = y$ is $(x, y) \to (y, x)$.
- The coordinate rule for the rotation by 90° is $(x, y) \to (y, x)$.
- The coordinate rule for the rotation by 180° is $(x, y) \to (-x, -y)$.
- The coordinate rule for the dilation with center the origin and factor r is $(x, y) \to (rx, ry)$.