Chapter 4
Simultaneous Linear Equations

Section 4.1: Understanding Solutions of Simultaneous Linear Equations

Analyze and solve pairs of simultaneous linear equations. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously. 8.EE.8a

Simultaneous linear equations refers to a pair of equations of the form $Ax + By = C$, where A, B, C are specific numbers, positive or negative. To say they are simultaneous is to ask: for what, if any, values substituted for the variables (x and y) are the equations both true at the same time? Those pairs of values are the *solutions* of the simultaneous equations. To illustrate: $x + 2y = 10$, $x - 3y = 0$ is a pair of equations, describing two relations between the variables x and y. If the context requires them to both be true, they are simultaneous. A solution is $x = 6$, $y = 2$, because that substitution makes both statements true. In this chapter, we want to explore procedures, both algebraic and graphical, to determine the solutions of simultaneous linear equations.

Example 1.

$x = 21$, $y = 17$ is a pair of simultaneous linear equations. Clearly there is only one solution, namely $x = 21$, $y = 17$. Suppose I want to disguise this: these are the ages of my older siblings, and I am asked for their ages. OK, I'll say: the sum of their ages is 38. The inquisitor is not satisfied: there are many pairs of numbers whose sum is 38. OK, I add the information that the difference in their ages is 4. Now, can the inquisitor determine their ages? If he lets them be x and y, he can now write down the two pieces of information algebraically: "the sum of their ages is 38" becomes $x + y = 38$. "The difference in their ages is 4" becomes $x - y = 4$. The two equations

$$x + y = 38, \quad x - y = 4$$

are a pair of simultaneous linear equations, for which the actual ages of my siblings are a solution. But is this enough information for the inquisitor to find the solution? If I know the sum and difference of two numbers, do we know the numbers? The answer to this is "yes," and to solve the problem we must discover how to get from the sum and difference of two numbers back to the originals.

Example 2.

Let's give another example, in the form of a game, that illustrates the same process:

1. Pick two numbers.
2. Double one and add the other. Tell me the result.
3. Now exchange the numbers and do the same. Tell me the result.

The numbers I received are 30 and 27. After a second, I say "Your two numbers were 8 and 11."

How did I get the answer so fast? Let's investigate this by analyzing the process. In Examples 1 and 2, we start with a particular pair of numbers. Then we perform algebraic operations on them: in the first problem we added the two numbers and then subtracted the two numbers. From $x = 21$, $y = 17$ we went to $x + y = 38$, $x - y = 4$. The challenge to the inquisitor is to find a way to go back. Maybe he should add these two equations:

$$x + y = 38, \quad x - y = 4; \quad \text{adding we get} \quad (x + y) + (x - y) = 38 + 4,$$

which simplifies to $2x = 42$, so we've found x: $x = 21$. Now let's subtract the two equations:

$$x + y = 38, \quad x - y = 4; \quad \text{subtracting we get} \quad (x + y) - (x - y) = 38 - 4,$$

which simplifies to $2y = 34$, so we have also found y: $y = 17$.

Let's analyze the second example in the same way. The two numbers picked are x and y. In step 2; we calculate $2x + y$ and inform the questioner that $2x + y = 30$. Then, in step 3, we form $2y + x$ and assert its value is 27. So the questioner knows that

$$2x + y = 30, \quad 2y + x = 27$$

Let's now do the same with this information: Add what we know:

$$(2x + y) + (2y + x) = 57,$$

which simplifies to $3(x + y) = 57$, or $x + y = 19$. Now subtract what we know:

$$(2x + y) - (2y + x) = 30 - 27$$

or $x - y = 3$. Now we know the sum and differences of the two numbers, so we can apply the technique of eExample 1 to obtain $x = 11$, $y = 8$.

The point of these examples is to see how to get from one pair of simultaneous equations to another, so that the solution set is the same. In one direction, when we want to disguise the numbers, we continue this process until we have sufficiently confounded the subject. In the other direction we select operations that unscramble that information. The tools we employ are these:

a. Add equals to equals ($x = 21$, $y = 17$ became $x + y = 38$);

b. Subtract equals from equals ($x = 21$, $y = 17$ became $x - y = 4$);

c. Multiply equals by a nonzero number (in the case of $2y = 34$,we multiply both sides by $1/2$).

Using these operations we can get from one pair of simultaneous equations to another pair so that the solution for the two pairs of equations does not change. Note that it is important, since we have two unknowns that we must have, at every stage, two equations. If I have in mind the numbers 17 and 21, and I tell you that the sum is 38, you do not have enough information to find the numbers. I either have to tell you that one of the numbers is 17 (or 21), or I have to give you another piece of scrambled information, such as: the difference of the two numbers is 4.

The point is not that we apply these operations at random, but we do so in order to reach our objectives. As we shall see in the next section, it is the form of our given information and the actual known numbers that show us the operations to use. There is one last tool for solving, that of substitution:

d. Replace an expression in one equation by an equal expression obtained from the other equation.

Example 3.

Lovasz has 5 marbles more than twice the number of marbles that Tonio has. Together they have 107 marbles. How many marbles does Lovasz have?

Solution. First we represent the unknown numbers by letters: Let L be the number of marbles that Lovasz has, and T the number of marbles that Tonio has. We are told that "Lovasz has 5 marbles more than twice Tonio's"; this translates to $L = 5 + 2T$. The second fact is that the sum of all the marbles is 107, so $L + T = 107$. The first equation tells us that L and $5 + 2T$ are the same number, so we can replace L by $5 + 2T$ in the second equation to get:

$$5 + 2T + T = 107.$$

From Chapter 1 we know how to solve a linear equation in one variable: combine like terms on the left and subtract 5 from both sides to get $3T = 102$, so $T = 34$: Tonio has 34 marbles. To find L, we turn to the first equation and replace T by 34, since we know these are equal, and we have $L = 5 + 2(34)$, so $L = 73$. We can use the second equation again to check this result: $73 + 34 = 107$.

Before going to the use of these operations to solve systems, we turn to the representation of this process by graphs.

Example 4.

Consider the linear equations $3x + y = 7$, $x + 3y = 5$. Graph both equations on a coordinate plane and find the coordinates (x, y) of the point of intersection.

Solution. The slope of the first line is -3, and of the second, $-1/3$. Since the slopes are not the same, the lines are not parallel, so they must intersect. Now, using a point on each line (for example, the y-intercepts $(0, 7)$ for the first line and $(0, 5/3)$ for the second, we graph the lines as in Figure 1.

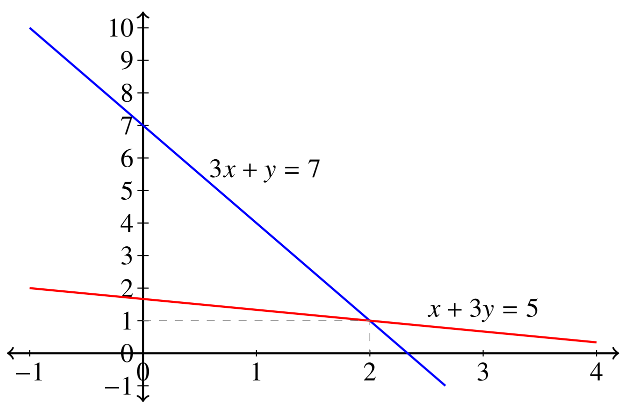


Figure 1

The figure shows the point of intersection to be $(2, 1)$. We should confirm that by checking that the substitution $x = 2$, $y = 1$ satisfies both equations:

$$3(2) + 1 = 7, \quad 2 + 3(1) = 5$$

In this example, we have graphed the given linear relations, and read off, from the graph, the coordinates $(2, 1)$ of the point of intersection. As this point lies on both lines, those coordinates ($x = 2, y = 1$) satisfy both equations. This is what it means to "solve the pair of simultaneous equations:"

Example 5.

Consider the linear equations $x - 2y = 8$, $2x + 5y = 34$. Graph each equation on the same grid with the same axes , and read off the coordinates (x, y) of the point of intersection.

Solution. Again we see that the two lines have different slopes ($1/2$ and $-2/5$), so the lines are not parallel and have a point of intersection. We draw the graphs of these equations (see Figure 2) and read off the coordinates of the point of intersection as $(12, 2)$. Since $(12, 2)$ lies on both lines, the values $x = 12$, $y = 2$ will satisfy both equations.

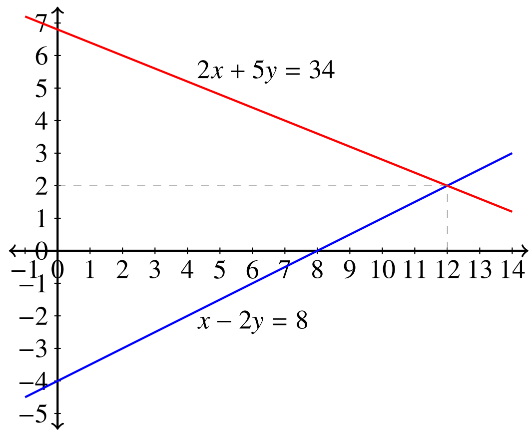


Figure 2

Furthermore, since there is only one point of intersection of two nonparallel lines, this is the unique solution.

What happens if the lines are parallel? We look at this case in the next example:

Example 6.

Consider the linear equations $2x + 5y = 10$, $4x + 10y = 40$. Graph each equation, and look for the point of intersection.

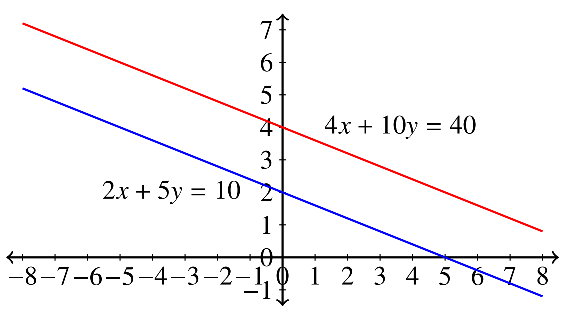


Figure 3

These equations provide graphs of two different lines (since they have different y-intercepts, $(0,2)$ and $(0,4)$) that are parallel, since they have the same slope $(-2/5)$. In particular, there is no point of intersection, which tells us that there are no values $x = a$, $y = b$ that satisfy both equations, The graph of these lines (Figure 3) confirms this.

Example 7.

Now, consider the linear equations $2x + 5y = 20$, $4x + 10y = 40$. The graphs of these equations are the same, since the lines they describe have the same slope and the same y-intercept.

In fact, if we put these equations into slope-intercept form, they both simplify to $y = -(\frac{2}{5})x + 4$. In this case, there are infinitely many solutions to the pair of equations, since the coordinates of every point on the line satisfies both equations. Notice that one of the original equations is a multiple of the other (divide both sides of the second equation by 2), so they are equivalent expressions. This will always be the case when a pair of simultaneous linear equations has more than one solution.

Summary: Given a pair of linear equations, there are three possibilities for simultaneous solutions:

1. The rate of change of y with respect to x is different for the two equations. In this case the graphs of the equations are lines with different slopes so are nonparallel, and intersect in a point. The coordinates of this point give the unique solution of the pair of equations.
2. The rate of change of y with respect to x is the same for the two equations, but they have different intercepts. In this case, the equations define lines with the same slope and thus parallel. If the lines are different, there is no solution to the simultaneous equations.
3. The rate of change of y with respect to x is the same for the two equations, and the equations define the same line. In this case, the coordinates of any point on the line gives a solution for the pair of equations.

In short, if two lines have different slopes (are not parallel) then there is a (single) point of intersection. If two lines have the same slope (are parallel), then either there is no solution, or they are the same line and there are many solutions.

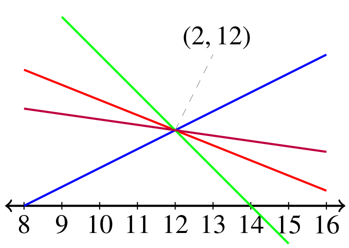


Figure 4

Take a look at the graphical implications of the operations on simultaneous equations described above. Start with the equations of Example 5: $x - 2y = 8$, $2x + 5y = 34$. The graphs are shown in Figure 2. Add the two given equations, getting $3x + 3y = 42$. If we subtract the first from the second, we get $x + 7y = 26$. Put the graphs of these two additional equations onto the Figure 2, resulting in Figure 4. The green and purple lines are the graphs of these new equations. Note that the graphs go through the same point; that is, the solution for the original pair of equations is also the solution for the new set of equations. So, we may now proceed to solve using the new set of equations. Our objective is to manipulate the equations with these operations so that we end with equations whose graphs are horizontal and vertical lines (in the case the graphs of $x = 12$ and $y = 2$).

Section 4.2: Solving Simultaneous Linear Equations Algebraically

Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. For example, $3x + 2y = 5$ and $3x + 2y = 6$ have no solution because $3x + 2y$ cannot simultaneously be 5 and 6. 8.EE.8b

Method of Substitution

This is a straightforward method, and is to be used when one of the pair of simultaneous linear equations expresses one variable (say y) in terms of the other (say x) . Then we can replace the y in the other equation by the expression in x, and obtain a new equation with only one variable. Example 3 above illustrates this method; here we further develop the idea.

Example 8.

I went to the grocery store and bought 15 pounds of grapefruit and oranges. I bought 3 fewer pounds of grapefruit than oranges. How many pounds of each fruit did I buy?

Solution. Pick letters to represent the number of pounds of grapefruit and oranges bought. I'll pick G for grapefruit, and R for oranges (since O is not a convenient symbol, being so close to the symbol for zero). The first statement says that $G + R = 15$. The second statement says that $G = R - 3$. So we can replace G in the first equation by $R - 3$, giving us $R - 3 + R = 15$. This simplifies to $2R = 18$, so I bought 9 pounds of oranges. Using the second statement, we see that I bought 6 pounds of grapefruit.

Example 9.

Two numbers (x, y) are related by the equation $5x - y = 17$. We also know that y is a function of x, given by $y = 2x + 4$. Find the solution.

Solution. Substituting, we get $5x - (2x + 4) = 17$. This we can solve for x: the equation becomes $3x - 4 = 17$, leading to $3x = 21$, and so $x = 7$. Now we use the functional relation between y and x to find y; $y = 2(7) + 4$, so $y = 18$.

Example 10.

Sometimes a little work is needed to write one of the variables in terms of the other. For example, solve

$$\begin{aligned} 2x + y &= 12 \\ 2x - 3y &= 4 \end{aligned}$$

Solution. We can rewrite the first equation as $y = 12 - 2x$, and then replace y in the second by this expression:

$$2x - 3(12 - 2x) = 4$$

Now we solve this equation for x, first simplifying to get $2x - 36 + 6x = 4$, or $8x = 40$, so that $x = 5$. The corresponding y is $y = 12 - 2(5) = 2$. Note another way to solve the equation: subtract the second equation from the first to get $4y = 8$, and thus $y = 2$.

Example 11.

Another day I went to the store and spent \$26.25 for 15 pounds of grapefruits and oranges. The grapefruit cost \$1.25 per pound and the oranges cost \$2.00 per pound. How many grapefruits and oranges did I buy?

Solution. Using the same letters as in Example 7, the first equation tells us that $G+R=15$. G pounds of grapefruit cost me \$1.25(G), and R pounds of oranges cost me \$2.00(R). The sum is \$26.25, giving me the equation

$$1.25G + 2R = 26.25$$

The first relation tells me that $R = 15 - G$, so I can replace R in the second equation by $15 - G$, giving me

$$1.25G + 2(15 - G) = 26.25$$

This becomes $1.25G + 30 - 2G = 26.25$, which simplifies to $.75G = 3.75$, so $G = 5$. Then, returning to the relation $G + R = 15$, we see that $R = 10$. I bought 5 pounds of grapefruit and 10 pounds of oranges.

Some of the choices made in this problem were arbitrary: to begin with, We could have solved for G in terms of R ($G = 15 - R$), and then written the equation in R:

$$1.25(15 - R) + 2R = 26.25$$

but it is better to have to multiply 2 and 15 instead of 1.25 and 15. Nevertheless the result would have been the same. And again, when we came to $.75G = 3.75$; we could change to fractions to get

$$\frac{3}{4}G = \frac{15}{4}$$

from which we see directly that $G = 5$. In almost all cases such choices have to be made, and should be made on the basis of making one's work as simple as possible.

The Substitution Algorithm is

1. Rewrite one of the equation so as to express one variable in term of the other;
2. Substitute that expression for that variable in the other equation;
3. Solve for the second variable, and put that value in the first equation to find the solution.

Method of Elimination

For many pairs of linear equations to be solved simultaneously, there are ways to find the solutions that are easier than trying to express one variable in terms of the other. Here we will describe a method that is the default method for most computational programs.

Example 12.

Solve

$$3x + 2y = 12$$
$$2x + 2y = 10$$

Solution. If we look carefully at the equations, we see that, on the left side, the difference between the first and the second expressions is x, and the difference on the right side is 2. Since we can add and subtract equations without changing the solution set, we thus choose to subtract the second equation from the first to get $x = 2$. Putting that value in either equation gives us an equation in the single unknown y, and we conclude from either that $y = 3$.

Example 13.

Solve $6x + 2y = 20$, $3x - y = 2$.

Solution. From the previous example, we learned that if the coefficient of y is the same, we can combine the equations so as to eliminate y, then solve for x, and use that known value for x in one of the preceding equations to solve for y. We can arrange this by multiplying both sides of the second equation by 2, giving us the pair of equations

$$6x + 2y = 20$$
$$6x - 2y = 4$$

If we add these equations y subtracts put and we get $12x = 24$, from which we find that $x = 2$. Now putting this value of x into either equation, gives us $y = 4$, so the complete solution is $(2, 4)$.

We could also subtract the equations to get $4y = 16$, giving us the value $y = 4$; upon substitution of this value in either equation, we get $x = 2$.

Example 14.

Solve

$$\frac{3}{5}x + \frac{1}{3}y = 12$$
$$\frac{1}{3}x + \frac{1}{2}y = 18$$

We include this example to illustrate that the complexity of the numbers involved should not be a deterrent to applying these methods, but one must be careful. There are various ways of solving this problem (besides feeding the data into a computer program) that make the arithmetic manageable. First, we might clear of fractions by multiplying the first equation by 15 and the second by 6 to get:

$$\begin{aligned} 9x + 5y &= 180 \\ 2x + 3y &= 48 \end{aligned}$$

Now multiply the first equation by 3 and the second by 5 so as to get the coefficients of y the same:

$$\begin{aligned} 27x + 15y &= 540 \\ 10x + 15y &= 240 \end{aligned}$$

Subtracting the second from the first produces $17x = 300$, so that $x = 300/17$. We can now put this value of x in any of the preceding equations to solve for y. Let's use the equation $2x + 3y = 48$. We get

$$2\left(\frac{300}{17}\right) + 3y = 48$$

leading to

$$3y = \frac{48 \times 17 - 600}{17}$$

or $3y = 216/17$, giving us the value $72/17$ for y.

We can avoid the large multiplications and divisions by first noting that if we multiply the original first equation by $3/2$, we make the coefficients of y the same:

$$\begin{aligned} \frac{9}{10}x + \frac{1}{2}y &= 18 \\ \frac{1}{3}x + \frac{1}{2}y &= 8 \end{aligned}$$

Now subtract the second from the first to eliminate y and get

$$\left(\frac{9}{10} - \frac{1}{3}\right)x = 10$$

Multiply by 30 to clear of fractions to get $(27 - 10)x = 300$, which is not the same equation for x we have above, so proceed in the same way.

One may be tempted to switch to decimals to get

$$\begin{aligned} 0.6x + 0.33y &= 12 \\ 0.33x + 0.5y &= 8 \end{aligned}$$

and then use a calculator for the computations to follow. However, one has to be careful: 0.33 is not $1/3$, but an approximation of $1/3$. As one goes through calculations, the error in the approximation tends to grow, often to the extent to make the end result untrustworthy. So, for example, to achieve accuracy within 2 decimal points, one should start with the approximation 0.3333 for $1/3$, making the calculations that much more difficult.

We summarize this procedure in the following set of rules. Keep in mind that in many of the examples above, and in the problems for discussion and homework, the particular numerical coefficients give a clue on how to proceed. In real-life problems we do not have that luxury: experimentally determined numerical coefficients are hardly ever so convenient.

The Elimination Algorithm is

1. Multiply the equations by nonzero numbers so that the coefficients of one of the unknowns are the same;
2. Take the difference (or sum) of the equations to obtain a new equation in just one unknown;
3. Solve for that unknown, then substitute that value in one of the original equations to solve for the other unknown.

Comments:

1. The first step is to arrange for the coefficients in the two equations of one of the variables to be the same. There can be many ways of doing this; they are all valid. For example, for the pair

$$\begin{aligned} 2x + 4y &= 12.30 \\ x + 5y &= 13.20 \end{aligned}$$

we could have multiplied the first equation by 5 and the second by 4 to obtain:

$$\begin{aligned} 10x + 20y &= 61.50 \\ 4x + 20y &= 52.80 \end{aligned}$$

Now the difference leads to $6x = 8.70$, and $x = 1.45$. It worked; nevertheless, a good rule to follow is this: look for the simplest multipliers to use. Here it would have been to multiply the second equation by 2 so as to eliminate x.

2. The second step suggests that the elimination may involve taking the sum, rather than the difference. For example:

$$\begin{aligned} 2x + 6y &= 38 \\ x - 3y &= 11 \end{aligned}$$

Step 1 suggests multiplying the second equation by 2 to obtain

$$2x + 6y = 38$$
$$2x - 6y = 22$$

Now, adding the equations will eliminate y and we get $4x = 60$, so $x = 15$. Notice that, if we took the difference, we get $12y = 16$, so $y = 4/3$.

3. When we eliminate one of the variables, suppose both disappear? Consider the pair of equations:

$$2x + y = 7$$
$$4x + 2y = 4$$

Following the algorithm, we multiply the first equation by 2 to get:

$$4x + 2y = 14$$
$$4x + 2y = 4$$

Subtracting the second equation from the first gives the equation $0x + 0y = 10$. Since there are no values of x and y that can make that statement true, the same is true for the original pair: there are no solutions. Notice that the slope of both lines is $-1/2$; that is, the lines are parallel. So, this corresponds to the graphical situation where the lines never cross. Similarly, if we consider this pair of equations:

$$2x + y = 7$$
$$4x + 2y = 14$$

and follow the rules, we end up with $0 = 0$, which is a true statement, but not a very informative one. What we are observing is that both equations define the same line, since the second equation is just double the first.

4. The choice of method to use is up to the solver, and depends upon the coefficients of the equations, just pick the method that is easier. For example, consider the pair of equations

$$3x + 7y = 18, \quad y = 6 + 3x$$

If we apply the elimination method, we first have to rewrite the second equation as $-3x + y = 6$, and then proceed. But, why? Our procedure is to isolate one of the variables and the second equation has done that for us. So, we can go right into the the replacement step, and substitute the value of y in terms of x given by the second equation, into the first, to get

$$3x + 7(6 + 3x) = 18$$

which simplifies to $24x + 42 = 18$, leading to $24x = -24$, or $x = -1$. Substituting that value into the other equation gives us $-3 + 7y = 18$, or $y = 3$.

Example 15.

Solve the pair of equations:

$$\begin{aligned} x + y &= 10 \\ 11x + 8y &= 92 \end{aligned}$$

The first equation tells us that $y = 10 - x$; substituting that in the second gives

$$11x + 8(10 - x) = 92$$

which we can now solve for x. We get $11x + 80 - 8x = 92$, which simplifies to $3x = 12$, with the result that $x = 4$. Now substitute that in the first equation to find that $y = 6$. Try the method of elimination, to compare the difficulty of both methods.

Example 16.

In the next example, it is not so clear at first which is the more direct method:

$$\begin{aligned} 4x + y &= 10 \\ 13x + 11y &= 79 \end{aligned}$$

Looking at the two equations, the method of elimination suggests multiplying the first equation by 11, leading to rather large and unwieldy numbers to work with. On the other hand, the first equation easily transforms into the equation $y = 10 - 4x$. Substituting $10 - 4x$ for y in the second equation gives us

$$\begin{aligned} 13x + 11(10 - 4x) &= 79 \\ 13x + 110 - 44x &= 79 \\ -31x &= -31 \end{aligned}$$

from which we get $x = 1$. Now substituting that in the first equation gives us $4 + y = 10$, or $y = 6$.

Don't forget that you have to use both equations! So, for example, if you solve the first equation for y in terms of x, substitute that expression in the second equation, not the first. If you substitute in the first, you get a true, but not very useful equation.

Example 17.

In each of the following pairs of simultaneous equations, determine the preferable method (*elimination* (E) or *substitution* (S)) for solving the system.

a. $y = 4$	**b.** $x = y - 1$	**c.** $x - 5y = 7$	**d.** $y = 2x - 1$	**e.** $9y = x - 1$
$y = 6x - 1$	$2x - 3y = 10$	$6x + 3y = 5$	$4x + y = 7$	$9y = x + 1$

Solution.

- **a.** Substitute 4 for y in the second equation. This seems more direct than elimination, by subtracting the first equation from the second, but that will also surely work.

- **b.** Substitute $y - 1$ for x in the second equation.

- **c.** Eliminate x by subtracting six times the first equation from the second.

- **d.** Substitute $2x - 1$ for y in the second equation.

- **e.** Set $x - 1$ equal to $x + 1$ since both are equal to $9y$. Notice that there are no solutions, since $x - 1 = x + 1$ has no solution.

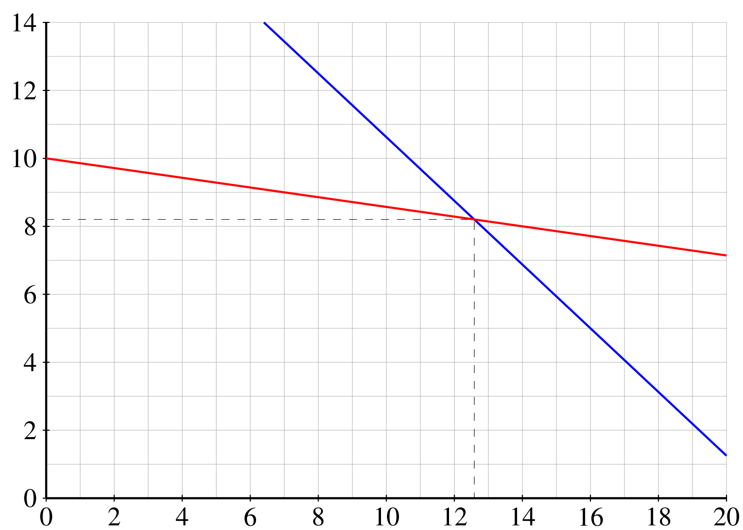
It should be noted that the choice of method is one of convenience, and not necessity. Elimination always works, and in fact that is the method used in computer programs.

Example 18.

In many contexts the coefficients will not work out well (as in Example 14 above), and we may be satisfied with an estimate of the solution. In such cases it is advisable to turn to graphical solutions. In fact, graphical solutions can only provide approximate solutions, unless the point in question is at the intersection of gridlines. Furthermore, the estimate is only as good as the grid lines of fine. Let's work an example for which we will be satisfied with an estimate within one decimal point. The equations are:

$$\frac{3}{4}x + \frac{4}{5}y = 16, \qquad \frac{1}{10}x + \frac{7}{10}y = 7$$

In Figure 5 we have graphed the lines corresponding to these equations, from which we can read the approximate solution $x = 12.6, y = 8.2$.



$\frac{3}{4}x + \frac{4}{5}y = 16$ $\quad \frac{1}{10}x - \frac{7}{10}y = 7$

Figure 5

Example 19.

It is very helpful to use a spreadsheet, like Excel, to find graphical solutions. On the spreadsheet, we can calculate two points on each line, and then graph the lines on the same grid. For the pair of equations

$$6x + 5y = 31$$
$$5x - 3y = 0$$

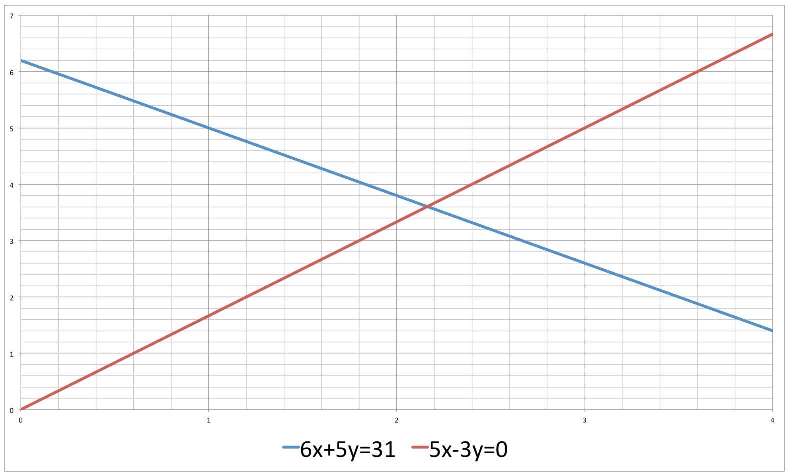


Figure 6

From the graph (see Figure 6) we estimate the coordinates of the point of intersection to be $(2.2, 3.6)$.

Solving Real World Problems using Systems

Solve real-world and mathematical problems leading to two linear equations in two variables. 8.EE.8c

Example 20.

Let's return to grapefruits and oranges (Example 11). Joanne and Rudy shop at the same store. Joanne bought 6 lbs of grapefruit and 4 lbs of oranges, and spent \$8.22. Rudy bought 6 lbs of grapefruit and 5 lbs of oranges and spent \$9.09. What is the cost of a pound of oranges? How much is a pound of grapefruit?

It is always a good idea to study the problem carefully, looking for clues for solving. For example, in this case we see that Rudy bought the same amount of grapefruit as Joanne, and one more pound of oranges than Joanne, and spent \$.87 more than Joanne. So, we conclude that a pound of oranges costs \$.87.

Now, Rudy spent \$9.09, of which $5 \times .87 = 4.35$ was on oranges. Thus he spent $9.09 - 4.35 = 4.74$ on 6 lbs of grapefruit , so each pound of grapefruit is worth $(4.74)/6 = .79$.

In general we may not be so lucky as to see a shortcut right away, so it is always a good idea to apply the rules for solving equations developed in Chapter 1. First: what do we want to find out? - the answer is the cost of a pound of grapefruit, and the cost of a pound of oranges. Let's call those unknowns g and r. (It might be good to point out that these are not the same G and R as in example 11: there the issue was of "pounds of fruit," here it is of "cost of fruit." We have chosen lower case letters to avoid this confusion). Now look at Joanne's purchase: 6 lbs of grapefruit at G per pound, and 4 lbs of oranges at R per pound. This totals to \$8.22, giving us the equation:

$$6g + 4r = 8.22$$

Do the same with Rudy's purchase, to get $6g + 5r = 9.09$. Using the general rules of elimination developed above, we subtract the first equation from the second to get $g = .87$, and continuing as above, we'll find $r = .79$.

Example 21.

Tickets to the local basketball arena cost \$54 for lower bowl seats and \$20 for upper bowl seats. The Math department purchased 123 tickets for their faculty and students; that purchase cost \$4262. How many of each type of ticket did they purchase?

Solution. The unknowns are the number of tickets at each level, so we set L as the number of lower bowl tickets, and U as the number of upper bowl tickets. So $L + U = 123$. The cost of lower bowl tickets is $54L$, and the cost of upper bowl tickets is $20U$. This gives us the pair of equations to solve:

$$\begin{aligned} L + U &= 123 \\ 54L + 20U &= 4262 \end{aligned}$$

We can write the first equation as $U = 123 - L$ and substitute that value of U in the second equation. Solving, we find that there are 53 lower bowl tickets and 70 upper bowl tickets. To confirm that this is the correct answer we check that $54(53) + 20(70) = 4262$.

Example 22.

Alfredo and Juanita also shop at a store in the same chain, but one that is eight states away. Joanne bought 6 lbs of grapefruit and 3 lbs of oranges, and spent \$10.77. Alfredo bought 3 pounds of grapefruit and 2 pounds of oranges and spent \$5.94. What is the cost of a pound of oranges? What is the cost of a pound of grapefruit?

Here we can't just look at the difference in the purchases: it tells us 3 lbs of grapefruit and 1 lb of oranges costs \$4.83, which is not of much help. But if we double Alfredo's purchase, we can say that 6 lbs of grapefruit and 4 lbs of oranges costs $2 \times 5.94 = 11.88$. This gives us the pair of equations (using the same meaning of g and r as in the preceding example), but eight states away:

$$\begin{aligned} 6g + 3r &= 10.77 \\ 6g + 4r &= 11.88 \end{aligned}$$

Now the difference on one side is 1 pound of oranges, and on the other, \$1.11. The cost of one pound of oranges is \$1.11. Let's go back to one of the original equations to find the cost of grapefruit. Alfredo spend \$2.22 on the 2 lbs of oranges, and $\$5.94 - 2.22 = 3.27$ on 3 lbs of grapefruit . Thus one pound of grapefruit costs $3.27/3 = 1.09$.

Extension

Example 23.

The Daybreak High School has 630 students. The boys outnumber the girls in the ratio 11 : 10. How many boys and how many girls are there?

Solution. There are B boys and G girls at Daybreak. We are told that $B+G = 630$ and $B : G = 11 : 10$. Turning the rato into an equation gives us

$$\frac{B}{G} = \frac{11}{10} \qquad \text{or} \quad B = \frac{11}{10}G\ .$$

Substitute that expression for B into the equation $B + G = 630$ and solve for G. We find $B = 330,\ G = 300$.

An easier way to solve the problem is to make better use of the information that we are told how many students there are. So, the ratio of girls to students is 10 : 21; that is $G : 630 = 10 : 21$, which gives the equation:

$$\frac{G}{630} = \frac{10}{21}$$

leading to $G = 300$ right away.

Example 24.

Each day, ferry companies A and B cross the straights of Gibraltar connecting the Spanish port of Tarifa with Tangier and Ceuta. Company A makes 4 round trips to Tangier and 3 to Ceuta, logging 332 miles. Company B makes 2 round trips to Tangier and 4 to Ceuta, logging 302 miles. What are the distances of Tarifa from Tangier and Tarifa from Ceuta?

Solution. Let T represent the distance in miles of Tarifa from Tangier, and C, the distance between Tarifa and Ceuta. Then a round trip from Tarifa to Tangier is $2T$ miles and a round trip from Tarifa to Ceuta is $2C$ miles. Company A makes 4 round trips to Tangier - that logs $4(2T)$ miles - and makes 3 round trips to Ceuta -that logs $3(2C)$ miles. The sum of the distances of all these trips is 332 miles, giving us the equation

$$4(2T) + 3(2C) = 332$$

Applying the same reasoning for company B leads to the equation

$$2(2T) + 4(2C) = 302$$

So, our task is to find the values of T and C that simultaneously satisfies the two equations

$$8T + 6C = 332, \quad 4T + 8C = 302$$

Now solve, to find the answers: Tarifa is 21.1 miles from Tangier, and Ceuta is 27.2 miles from Tangier.

Example 25.

Lisa is interested in discovering the rate of gas consumption of her new car, both in city miles and freeway miles. She selects two weeks in which she is driving in the city during the weekdays, and goes on a road trip on the weekend. The following table shows the miles she logged:

	City	Freeway
Week 1	131	210
Week 2	180	120

In each week she consumed 14.3 gallons. In miles per gallon, compute her rates of consumption both in city miles and in freeway miles.

SOLUTION. Lisa wants to know the values of "city miles per gallon" and "freeway miles per gallon." But, while the units of the rows in the table are miles, the information she has is that the sums across the rows in this table are in *gallons*. The equations she has to write down for each week are of the form:

$$(*) \qquad \text{gallons of city driving in the week} + \text{gallons of freeway driving in the week} = 14.3\ .$$

Remembering that

$$(**) \qquad \text{gallons} = \text{miles}\frac{\text{gallons}}{\text{miles}}\ ,$$

she realizes that can convert the data of the table into equations involving gallons by choosing as the variables "gallons per miles." Now the equation (*) becomes

$$(*) \qquad \text{city miles}\frac{\text{city gallons}}{\text{city miles}} + \text{freeway miles}\frac{\text{freeway gallons}}{\text{freeway miles}} = 14.3\ .$$

So, she labels the variables in which she is interested as C = city gallons per mile, and F = = freeway gallons per mile. Uusing these variables the rows lead to this system of equations:

$$\begin{aligned} 131C + 210F &= 14.3 \\ 180C + 120F &= 14.3 \end{aligned}$$

Noticing that the numbers on the right side of each equation is the same, it is tempting to subtract the second from the first to get:

$$-49C + 90F = 0$$

or $F = (49)/(90)C$, which is already a startling fact: freeway driving is about $5/9$ the cost (in terms of fuel used) of city driving. But to get the original problem, we start by replacing F in one of the original equations by this expression in terms of C, and then solve for C. We get $C = 14.3/245 = 0.058$. If we substitute this into either of the original equations, we can solve for F , to find that $F = 0.031$. Now, we get back to the original goal of the problem: to find the rate of fuel used with respect to miles for city driving and for freeway driving. These are the reciprocals of the values of C and F. Calculating those reciprocals, we find that, for Lisa's new automobile, city driving gets 17.13 city miles per gallon, and 32.25 miles per gallon on the freeway.